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# Solutions of the Yang–Baxter equation with extra non-additive parameters: II. $U_q(gl(m|n))$

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**Abstract.** The type-I quantum superalgebras are known to admit non-trivial one-parameter families of inequivalent finite dimensional irreps, even for generic $q$. We apply the recently developed technique to construct new solutions to the quantum Yang–Baxter equation associated with the one-parameter family of irreps of $U_q(gl(m|n))$, thus obtaining $R$-matrices that depend not only on a spectral parameter but in addition on further continuous parameters. These extra parameters enter the Yang–Baxter equation in a similar way to the spectral parameter but in a non-additive form.

In [1], we developed a systematic method for constructing $R$-matrices (solutions of the quantum Yang–Baxter equation (QYBE)) associated with the multiplicity-free tensor product of *any* two affinizable irreps of a quantum algebra. This approach was applied and extended to quantum superalgebras in [2, 3]. For the type-I quantum superalgebra $U_q(gl(m|1))$ in particular, we were able to obtain $R$-matrices depending continuously on extra parameters, entering in a similar way to the spectral parameter but in a non-additive form. In this paper, we continue this study to construct new $R$-matrices for the type-I quantum superalgebra $U_q(gl(m|n))$ for any $m \geqslant n$.

The freedom of having extra continuous parameters in $R$-matrices opens up new and exciting possibilities. For example, in [4], by using the $R$-matrix associated with the one-parameter family of four-dimensional irreps of $U_q(gl(2|1))$, we derived a new exactly solvable lattice model of strongly correlated electrons on the unrestricted $4^L$-dimensional electronic Hilbert space $\otimes_{n=1}^{L} \mathbf{C}^4$ (where $L$ is the lattice length), which is a $gl(2|1)$ supersymmetric generalization of the Hubbard model with the Hubbard on-site interaction coupling coefficient related to the parameter carried by the four-dimensional irrep.

The origin of the extra parameters in our solutions are the parameters which are carried by the irreps themselves of the associated quantum superalgebra. As is well known [5], type-I superalgebras admit non-trivial one-parameter families of finite-dimensional irreps which deform to provide one-parameter families of finite-dimensional irreps of the corresponding type-I quantum superalgebras, for generic $q$ [6]. Note, however, that for quantum simple

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bosonic Lie algebras families of finite-dimensional representations are possible only when the deformation parameter $q$ is a root of unity. Therefore our solutions are not related to the chiral Potts model $R$-matrices which arise from quantum bosonic algebras for $q$ a root of unity only [7, 8].

Let us give a brief review of our general formalism formulated in [2, 3]. Let $G$ denote a simple Lie superalgebra of rank $r$ with generators $\{e_i, f_i, h_i\}$ and let $\alpha_i$ be its simple roots. Then the quantum superalgebra $U_q(G)$ can be defined with the structure of a $\mathbb{Z}_2$-graded quasi-triangular Hopf algebra. We will not give the full defining relations of $U_q(G)$ here, but simply mention that $U_q(G)$ has a coproduct structure given by

$$\Delta(q^{h_i/2}) = q^{h_i/2} \otimes q^{h_i/2} \qquad \Delta(a) = a \otimes q^{-h_i/2} + q^{h_i/2} \otimes a \qquad a = e_i, f_i. \tag{1}$$

The multiplication rule for the tensor product is defined for elements $a, b, c, d \in U_q(G)$ by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd) \tag{2}$$

where $[a] \in \mathbb{Z}_2$ denotes the degree of the element $a$.

Let $\pi_\alpha$ be a one-parameter family of irreps of $U_q(G)$ afforded by the irreducible module $V(\Lambda_\alpha)$ in such a way that the highest weight of the irrep depends on the parameter $\alpha$. Assume for any $\alpha$ that the irrep $\pi_\alpha$ is affinizable, i.e. it can be extended to an irrep of the corresponding quantum affine superalgebra $U_q(\hat{G})$. Consider an operator ($R$-matrix) $R(x|\alpha,\beta) \in \mathrm{End}(V(\Lambda_\alpha) \otimes V(\Lambda_\beta))$, where $x \in \mathbb{C}$ is the usual spectral parameter and $\pi_\alpha$, $\pi_\beta$ are two irreps from the one-parameter family. It has been shown by Jimbo [9] that a solution to the linear equations

$$\begin{aligned} &R(x|\alpha,\beta)\Delta^{\alpha\beta}(a) = \bar{\Delta}^{\alpha\beta}(a)R(x|\alpha,\beta) \qquad \forall\, a \in U_q(G) \\ &R(x|\alpha,\beta)\left(x\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}) + \pi_\alpha(q^{h_0/2}) \otimes \pi_\beta(e_0)\right) \\ &\qquad = \left(x\pi_\alpha(e_0) \otimes \pi_\beta(q^{h_0/2}) + \pi_\alpha(q^{-h_0/2}) \otimes \pi_\beta(e_0)\right) R(x|\alpha,\beta) \end{aligned} \tag{3}$$

satisfies the QYBE in the tensor product module $V(\Lambda_\alpha) \otimes V(\Lambda_\beta) \otimes V(\Lambda_\gamma)$ of three irreps from the one-parameter family

$$R_{12}(x|\alpha,\beta)R_{13}(xy|\alpha,\gamma)R_{23}(y|\beta,\gamma) = R_{23}(y|\beta,\gamma)R_{13}(xy|\alpha,\gamma)R_{12}(x|\alpha,\beta). \tag{4}$$

In the above, $\bar{\Delta} = T\cdot\Delta$, with $T$ the twist map defined by $T(a \otimes b) = (-1)^{[a][b]} b \otimes a$, $\forall\, a, b \in U_q(G)$ and $\Delta^{\alpha\beta}(a) = (\pi_\alpha \otimes \pi_\beta)\Delta(a)$; also, if $R(x|\alpha,\beta) = \sum_i \pi_\alpha(a_i) \otimes \pi_\beta(b_i)$, then $R_{12}(x|\alpha,\beta) = \sum_i \pi_\alpha(a_i) \otimes \pi_\beta(b_i) \otimes I$, etc. Jimbo also showed that the solution to (3) is unique, up to scalar functions. The multiplicative spectral parameter $x$ can be transformed into an additive spectral parameter $u$ by $x = \exp(u)$.

In all our equations we implicitly use the 'graded' multiplication rule of (2). Thus the $R$-matrix of a quantum superalgebra satisfies a 'graded' QYBE which, when written as an ordinary matrix equation, contains extra signs:

$$\begin{aligned} &(R(x|\alpha,\beta))_{ij}^{i'j'}\,(R(xy|\alpha,\gamma))_{i'k}^{i''k'}\,(R(y|\beta,\gamma))_{j'k'}^{j''k''}\,(-1)^{[i][j]+[k][i']+[k'][j']} \\ &\qquad = (R(y|\beta,\gamma))_{jk}^{j'k'}\,(R(xy|\alpha,\gamma))_{ik'}^{i'k''}\,(R(x|\alpha,\beta))_{i'j'}^{i''j''}\,(-1)^{[j][k]+[k'][i]+[j'][i']}. \end{aligned} \tag{5}$$

However, after a redefinition

$$\left(\tilde{R}(\cdot|\alpha,\beta)\right)_{ij}^{i'j'} = (R(\cdot|\alpha,\beta))_{ij}^{i'j'}\,(-1)^{[i][j]} \tag{6}$$

the signs disappear from the equation. Thus any solution of the 'graded' QYBE arising from the $R$-matrix of a quantum superalgebra provides also a solution of the standard QYBE after the redefinition in (6).

Introduce the graded permutation operator $P^{\alpha\beta}$ on the tensor product module $V(\Lambda_\alpha) \otimes V(\Lambda_\beta)$ such that

$$P^{\alpha\beta}(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]} v_\beta \otimes v_\alpha \qquad \forall\, v_\alpha \in V(\Lambda_\alpha) \quad v_\beta \in V(\Lambda_\beta) \tag{7}$$

and set

$$\check{R}(x|\alpha,\beta) = P^{\alpha\beta} R(x|\alpha,\beta). \tag{8}$$

Then (3) can be rewritten as

$$\begin{aligned}
&\check{R}(x|\alpha,\beta)\Delta^{\alpha\beta}(a) = \Delta^{\beta\alpha}(a)\check{R}(x|\alpha,\beta) \qquad \forall\, a \in U_q(G) \\
&\check{R}(x|\alpha,\beta)\left(x\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}) + \pi_\alpha(q^{h_0/2}) \otimes \pi_\beta(e_0)\right) \\
&\qquad = \left(\pi_\beta(e_0) \otimes \pi_\alpha(q^{-h_0/2}) + x\pi_\beta(q^{h_0/2}) \otimes \pi_\alpha(e_0)\right) \check{R}(x|\alpha,\beta)
\end{aligned} \tag{9}$$

and in terms of $\check{R}(x|\alpha,\beta)$ the QYBE becomes

$$\begin{aligned}
&(I \otimes \check{R}(x|\alpha,\beta))(\check{R}(xy|\alpha,\gamma) \otimes I)(I \otimes \check{R}(y|\beta,\gamma)) \\
&\qquad = (\check{R}(y|\beta,\gamma) \otimes I)(I \otimes \check{R}(xy|\alpha,\gamma))(\check{R}(x|\alpha,\beta) \otimes I)
\end{aligned} \tag{10}$$

both sides of which act from $V(\Lambda_\alpha) \otimes V(\Lambda_\beta) \otimes V(\Lambda_\gamma)$ to $V(\Lambda_\gamma) \otimes V(\Lambda_\beta) \otimes V(\Lambda_\alpha)$. Note that this equation, if written in matrix form, does not have extra signs. This is because the definition of the graded permutation operator in (7) includes the signs of (6). In what follows we will normalize the $R$-matrix $\check{R}(x|\alpha,\beta)$ in such a way that $\check{R}(x|\alpha,\beta)\check{R}(x^{-1}|\beta,\alpha) = I$, which in the literature is usually called the unitarity condition.

For three special values of $x$: $x = 0$, $x = \infty$ and $x = 1$, $\check{R}(\cdot|\alpha,\beta)$ satisfies the spectral-free, but extra non-additive-parameter-dependent QYBE:

$$\begin{aligned}
&(I \otimes \check{R}(\cdot|\alpha,\beta))(\check{R}(\cdot|\alpha,\gamma) \otimes I)(I \otimes \check{R}(\cdot|\beta,\gamma)) \\
&\qquad = (\check{R}(\cdot|\beta,\gamma) \otimes I)(I \otimes \check{R}(\cdot|\alpha,\gamma)(\check{R}(\cdot|\alpha,\beta) \otimes I).
\end{aligned} \tag{11}$$

In the case of a multiplicity-free tensor product decomposition

$$V(\Lambda_\alpha) \otimes V(\Lambda_\beta) = \bigoplus_\mu V(\mu) \tag{12}$$

where $\mu$ denotes a highest weight depending on the parameters $\alpha$ and $\beta$, the $\check{R}(0|\alpha,\beta)$, $\check{R}(\infty|\alpha,\beta)$ and $\check{R}(1|\alpha,\beta)$ may be obtained in the particularly simple form [1–3] for quantum bosonic algebras obtained in [10, 11]:

$$\check{R}(0|\alpha,\beta) = \sum_\mu \epsilon(\mu)\, q^{\frac{1}{2}[C(\mu)-C(\Lambda_\alpha)-C(\Lambda_\beta)]}\, P_\mu^{\alpha\beta} \tag{13}$$

$$\check{R}(\infty|\alpha,\beta) = \sum_\mu \epsilon(\mu)\, q^{-\frac{1}{2}[C(\mu)-C(\Lambda_\alpha)-C(\Lambda_\beta)]}\, P_\mu^{\alpha\beta} \tag{14}$$

$$\check{R}(1|\alpha,\beta) = \sum_\mu P_\mu^{\alpha\beta} \tag{15}$$

where $C(\Lambda) = (\Lambda, \Lambda + 2\rho)$ is the eigenvalue of the quadratic Casimir invariant of $G$ in the irrep with highest weight $\Lambda$, $\rho$ is the graded half-sum of positive roots of $G$ and $P_\mu^{\alpha\beta}$ : $V(\Lambda_\alpha) \otimes V(\Lambda_\beta) \to V(\mu) \subset V(\Lambda_\beta) \otimes V(\Lambda_\alpha)$ are the elementary intertwiners, i.e. $P_\mu^{\alpha\beta}\Delta^{\alpha\beta}(a) = \Delta^{\beta\alpha}(a)P_\mu^{\alpha\beta}$, $\epsilon(\mu)$ is the parity of $V(\mu)$ in $V(\Lambda_\alpha) \otimes V(\Lambda_\beta)$. Since in the present case $\alpha$, etc, are continuous parameters, the parities $\epsilon(\mu)$ can easily be worked out by examining the limit $\alpha \to \beta$.

The elementary intertwiners satisfy the relations

$$
\begin{aligned}
&\boldsymbol{P}_{\mu}^{\alpha\beta}\,\mathcal{P}_{\mu'}^{\alpha\beta}=\mathcal{P}_{\mu'}^{\beta\alpha}\,\boldsymbol{P}_{\mu}^{\alpha\beta}=\delta_{\mu\mu'}\boldsymbol{P}_{\mu}^{\alpha\beta}\\
&\boldsymbol{P}_{\mu}^{\beta\alpha}\,\boldsymbol{P}_{\mu'}^{\alpha\beta}=\delta_{\mu\mu'}\mathcal{P}_{\mu}^{\alpha\beta}
\end{aligned}
\tag{16}
$$

where the $\mathcal{P}_{\mu}^{\alpha\beta}:\ V(\Lambda_\alpha)\otimes V(\Lambda_\beta)\to V(\mu)\subset V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$ are projection operators satisfying

$$
\mathcal{P}_{\mu}^{\alpha\beta}\,\mathcal{P}_{\mu'}^{\alpha\beta}=\delta_{\nu\nu'}\mathcal{P}_{\nu}^{\alpha\beta}\qquad\qquad \sum_{\mu}\mathcal{P}_{\mu}^{\alpha\beta}=I.
\tag{17}
$$

Let $\{|e_i^\mu\rangle_{\alpha\otimes\beta}\}$ be an orthonormal basis for $V(\mu)$ in $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$. $V(\mu)$ is also embedded in $V(\Lambda_\beta)\otimes V(\Lambda_\alpha)$ through the opposite coproduct $\bar{\Delta}$. Let $\{|e_i^\mu\rangle_{\beta\otimes\alpha}\}$ be the corresponding orthonormal basis†. Using these bases the operators $\mathcal{P}_{\mu}^{\alpha\beta}$ and $\boldsymbol{P}_{\mu}^{\alpha\beta}$ can be expressed as

$$
\begin{aligned}
\mathcal{P}_{\mu}^{\alpha\beta}&=\sum_i \left|e_i^\mu\right\rangle_{\alpha\otimes\beta}\ {}_{\alpha\otimes\beta}\!\left\langle e_i^\mu\right|\\
\boldsymbol{P}_{\mu}^{\alpha\beta}&=\sum_i \left|e_i^\mu\right\rangle_{\beta\otimes\alpha}\ {}_{\alpha\otimes\beta}\!\left\langle e_i^\mu\right|\,.
\end{aligned}
\tag{18}
$$

The most general $\check{R}(x|\alpha,\beta)$ satisfying the first equation in (9) may be written in the form

$$
\check{R}(x|\alpha,\beta)=\sum_{V(\mu)\in V(\Lambda_\alpha)\otimes V(\Lambda_\beta)}\rho_\mu(x)\,\boldsymbol{P}_{\mu}^{\alpha\beta}
\tag{19}
$$

where the $\rho_\mu(x)$ are unknown functions depending on $x$, $q$ and the extra non-additive parameters $\alpha$ and $\beta$. It follows [1–3] from the second equation of (9) that if

$$
\mathcal{P}_{\mu}^{\alpha\beta}\left(\pi_\alpha(e_0)\otimes\pi_\beta(q^{-h_0/2})\right)\mathcal{P}_{\mu'}^{\alpha\beta}\neq 0
\tag{20}
$$

then the coefficients $\rho_\mu(x)$ in (19) are given recursively by

$$
\rho_\mu(x)=\rho_{\mu'}(x)\frac{q^{C(\mu)/2}+\epsilon(\mu)\epsilon(\mu')xq^{C(\mu')/2}}{xq^{C(\mu)/2}+\epsilon(\mu)\epsilon(\mu')q^{C(\mu')/2}}\qquad \forall\ \mu\neq\mu'.
\tag{21}
$$

$\epsilon(\mu)\epsilon(\mu')=-1$ always holds if (20) is satisfied. With the help of the notation

$$
\langle a\rangle\equiv\frac{1-xq^a}{x-q^a}
\tag{22}
$$

equation (21) then becomes

$$
\rho_\mu(x)=\left\langle\frac{C(\mu')-C(\mu)}{2}\right\rangle\rho_{\mu'}(x)\,.
\tag{23}
$$

We have a relation between the coefficients $\rho_\mu$ and $\rho_{\mu'}$ whenever the condition (20) is satisfied, i.e. whenever $\pi_\alpha(e_0)\otimes\pi_\beta(q^{-h_0/2})$ maps from the module $V(\mu')$ to the module $V(\mu)$. As a graphical aid [12] we introduce the tensor product graph.

*Definition 1.* The *tensor product graph* (TPG) associated with the tensor product $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$ is a graph whose vertices are the irreducible modules $V(\mu)$ appearing in the decomposition of $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$. There is an edge between a vertex $V(\mu)$ and a vertex $V(\mu')$ iff

$$
\mathcal{P}_{\mu}^{\alpha\beta}\left(\pi_\alpha(e_0)\otimes\pi_\beta(q^{-h_0/2})\right)\mathcal{P}_{\mu'}^{\alpha\beta}\neq 0.
\tag{24}
$$

† For the precise definition of this basis see appendix C of [1].

If $V(\Lambda_\alpha)$ and $V(\Lambda_\beta)$ are irreducible $U_q(G)$ modules then the TPG is always connected, i.e. every node is linked to every other node by a path of edges. This implies that *relations (23) are sufficient to determine all the coefficients $\rho_\mu(x)$ uniquely, up to an overall factor.* If the TPG is multiply connected, i.e. if there exist more than two paths between two nodes, then the relations overdetermine the coefficients, i.e. there are consistency conditions. However, because the existence of a solution to the Jimbo equations is guaranteed by the existence of the universal $R$-matrix, these consistency conditions will always be satisfied.

The straightforward but tedious and impractical way to determine the TPG is to work out explicitly the left-hand side of (24). It is much more practical to work instead with the following larger graph which is often enough to determine the coefficients $\rho_\mu(x)$.

*Definition 2.* The *extended tensor product graph* (ETPG) associated with the tensor product $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$ is a graph whose vertices are the irreducible modules $V(\mu)$ appearing in the decomposition of $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$. There is an edge between two vertices $V(\mu)$ and $V(\mu')$ iff

$$V(\mu')\subset V_{\mathrm{adj}}\otimes V(\mu) \qquad \text{and} \qquad \epsilon(\mu)\epsilon(\mu')=-1. \tag{25}$$

The condition in (25) is a necessary condition for (24) [12]. This means that every link contained in the TPG is contained also in the ETPG but the latter may contain more links. Only if the ETPG is a tree do we know that it is equal to the TPG. If we impose a relation (23) on the $\rho$'s for every link in the ETPG, we may be imposing too many relations and thus *may not always find a solution. If, however, we do find a solution, then this is the unique* correct solution which we would also have obtained from the TPG.

We now apply the above formalism to the one-parameter family of irreps of $U_q(gl(m|n))$, all irreps of which are known to be affinizable.

Choose $\{\varepsilon_i\}_{i=1}^m\bigcup\{\bar\varepsilon_j\}_{j=1}^n$ as a basis for the dual of the Cartan subalgebra of $gl(m|n)$ satisfying

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij} \qquad (\bar\varepsilon_i,\bar\varepsilon_j)=-\delta_{ij} \qquad (\varepsilon_i,\bar\varepsilon_j)=0\,. \tag{26}$$

Using this basis, any weight $\Lambda$ may written as

$$\Lambda\equiv(\Lambda_1,\cdots,\Lambda_m|\bar\Lambda_1,\cdots,\bar\Lambda_n)\equiv\sum_{i=1}^m\Lambda_i\varepsilon_i+\sum_{j=1}^n\bar\Lambda_j\bar\varepsilon_j \tag{27}$$

and the graded half sum $\rho$ of the positive roots of $gl(m|n)$ is

$$2\rho=\sum_{i=1}^m(m-n-2i+1)\varepsilon_i+\sum_{j=1}^n(m+n-2j+1)\bar\varepsilon_j\,. \tag{28}$$

We assume $m\geqslant n$ and for $0\leqslant k\leqslant mn$ we call a Young diagram $[\lambda]=[\lambda_1,\lambda_2,\cdots,\lambda_r]$, $\lambda_1\geqslant\lambda_2\cdots\geqslant\lambda_r\geqslant 0$ for the permutation group $S_k$ (i.e. $\lambda_1+\lambda_2+\cdots+\lambda_r=k$) *allowable*, if it has at most $n$ columns and $m$ rows; i.e. $r\leqslant m$, $\lambda_i\leqslant n$. Associated with each such Young diagram $[\lambda]$ we define a weight of $gl(m|n)$

$$\Lambda_{[\lambda]}=(\dot 0_{m-r},-\lambda_r,\cdots,-\lambda_1|\underbrace{r,\cdots,r}_{\lambda_r},\underbrace{r-1,\cdots,r-1}_{\lambda_{r-1}-\lambda_r},\cdots,\underbrace{1,\cdots,1}_{\lambda_1-\lambda_2},\underbrace{0,\cdots,0}_{n-\lambda_1}) \tag{29}$$

In what follows we will consider the one-parameter family of finite-dimensional irreducible $U_q(gl(m|n))$ modules $V(\Lambda_\alpha)$ with highest weights of the form $\Lambda_\alpha=(0,\cdots,0|\alpha,\cdots,\alpha)\equiv(\dot 0|\dot\alpha)$. These irreps $V(\Lambda_\alpha)$ are unitary of type I if $\alpha>n-1$ and unitary of type II if $\alpha<1-m$. Here we assume real $\alpha$ satisfying one of these conditions, in which case $V(\Lambda_\alpha)$ is also typically of dimension $2^{mn}$.

We have the following decomposition of $V(\Lambda_\alpha)$ into irreps of the even subalgebra $gl(m) \bigoplus gl(n)$:

$$V(\Lambda_\alpha) = \bigoplus_{k=0}^{mn} \bigoplus_{[\lambda]\in S_k}{}' V_0(\Lambda_{[\lambda]} + \Lambda_\alpha) \tag{30}$$

where the prime denotes summation over allowed $k$-box Young diagrams. Note that the index $k$ gives the $\mathbb{Z}$-graded level of the irrep concerned. Alternatively we may simply write

$$V(\Lambda_\alpha) = \bigoplus_{[\lambda]}{}' V_0(\Lambda_{[\lambda]} + \Lambda_\alpha)\,. \tag{31}$$

The number of boxes then gives the level. For $\Lambda_\alpha$, $\Lambda_\beta$ of the same type, we immediately deduce the tensor product decomposition

$$V(\Lambda_\alpha) \otimes V(\Lambda_\beta) = \bigoplus_{[\lambda]}{}' V(\Lambda_{[\lambda]} + \Lambda_{\alpha+\beta})\,. \tag{32}$$

The eigenvalue of the second order Casimir on the irrep $V(\Lambda_{[\lambda]} + \Lambda_{\alpha+\beta})$ can be shown to be

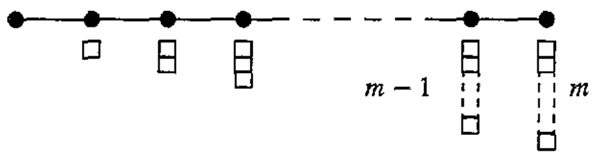
$$C([\lambda]) = 2\sum_{i=1}^{r} \lambda_i(\lambda_i + 1 - \alpha - \beta - 2i) - n(\alpha+\beta)(\alpha+\beta+m)\,. \tag{33}$$

Below we show that the ETPG corresponding to the above tensor product is always consistent and we derive an explicit expression for the eigenvalues $\rho_{[\lambda]}(x)$ for the $R$-matrix $\check{R}(x|\alpha,\beta)$ of $U_q(gl(m|n))$. It is instructive to first consider some examples.

*Example 1: $U_q(gl(m|1))$.* In this case we have the tensor product decomposition

$$\begin{aligned} V(\Lambda_\alpha) \otimes V(\Lambda_\beta) = {} & V(\lambda_{\alpha+\beta}) \oplus V(\dot{0}, -1|\alpha+\beta+1) \oplus V(\dot{0}, -1, -1|\alpha+\beta+2) \\ & \oplus \cdots \oplus V(-\dot{1}|\alpha+\beta+m)\,. \end{aligned} \tag{34}$$

In terms of Young diagrams the ETPG is




which is obviously consistent.

*Example 2: $U_q(gl(2|2))$.* The tensor product decomposition is

$$\begin{aligned} V(\Lambda_\alpha) \otimes V(\Lambda_\beta) = {} & V(\lambda_{\alpha+\beta}) \oplus V(0, -1|\alpha+\beta+1, \alpha+\beta) \\ & \oplus V(-1, -1|\alpha+\beta+2, \alpha+\beta) \oplus V(0, -2|\alpha+\beta+1, \alpha+\beta+1) \\ & \oplus V(-1, -2|\alpha+\beta+2, \alpha+\beta+1) \oplus V(-2, -2|\alpha+\beta+2, \alpha+\beta+2)\,. \end{aligned} \tag{35}$$

The ETPG in terms of Young diagrams is shown in figure 1($a$). From equation (33) for $C([\lambda])$ it can easily be deduced that

$$C\left(\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\right) - C\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) = C(\square\square) - C(\square) = -2(\alpha+\beta-2) \tag{36}$$

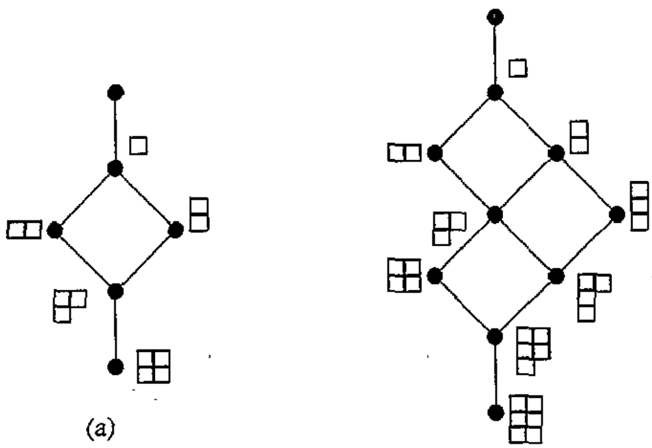


**Figure 1.** The ETPGs for $V(\Lambda_\alpha)\otimes V(\Lambda_\beta)$ in $(a)$ $U_q(gl(2|2))$ and $(b)$ $U_q(gl(3|2))$.

so that this diagram is consistent. For completeness we note that

$$
\begin{aligned}
&C(\cdot) = -2(\alpha+\beta)(\alpha+\beta+2) && C\left(\ydiagram{1}\right) = -2(\alpha+\beta)+C(\cdot)\\
&C\left(\ydiagram{2}\right) = -4(\alpha+\beta-1)+C(\cdot) && C\left(\ydiagram{1,1}\right) = -4(\alpha+\beta+1)+C(\cdot)\\
&C\left(\ydiagram{2,1}\right) = -6(\alpha+\beta)+C(\cdot) && C\left(\ydiagram{2,2}\right) = -8(\alpha+\beta)+C(\cdot).
\end{aligned}
\tag{37}
$$

*Example 3: $U_q(gl(3|2))$.* The ETPG is given in figure 1($b$). In this case equation (33) gives

$$
\begin{aligned}
C\left(\ydiagram{2,1}\right)-C\left(\ydiagram{1,1}\right) &= C\left(\ydiagram{2}\right)-C\left(\ydiagram{1}\right) = -2(\alpha+\beta-2)\\
C\left(\ydiagram{2,1,1}\right)-C\left(\ydiagram{1,1,1}\right) &= C\left(\ydiagram{2,1}\right)-C\left(\ydiagram{1,1}\right) = -2(\alpha+\beta-2)\\
C\left(\ydiagram{2,2,1}\right)-C\left(\ydiagram{2,1,1}\right) &= C\left(\ydiagram{2,2}\right)-C\left(\ydiagram{2,1}\right) = -2(\alpha+\beta).
\end{aligned}
\tag{38}
$$

So again all closed loops are consistent, leading to a consistent graph.

Now we return to the general case $U_q(gl(m|n))$. Corresponding to the tensor product decomposition (32), we note that given an (allowable) Young diagram $[\lambda]$, the number of boxes in $[\lambda]$ gives the level of the irrep $V([\lambda]+\Lambda_{\alpha+\beta})$ in the ETPG. We denote by $[\lambda+\Delta_r]$ the Young diagram obtained from $[\lambda]$ by increasing row $r$ by one box leaving the remaining rows unchanged.

Therefore in general we necessarily have closed loops of the form

$$
\begin{tikzcd}
 & {[\lambda]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[\lambda+\Delta_r]} \arrow[dr, dash] & & {[\lambda+\Delta_k]} \arrow[dl, dash] \\
 & {[\lambda+\Delta_r+\Delta_k]} &
\end{tikzcd}
\tag{39}
$$

and note that

$$C([\lambda+\Delta_r]) - C([\lambda]) = 2(\lambda_r+1)(\lambda_r+2-\alpha-\beta-2r) - 2\lambda_r(\lambda_r+1-\alpha-\beta-2r)$$
$$= 2(2\lambda_r+2-\alpha-\beta-2r)\,. \quad (40)$$

The important point, which is easily seen, is that (for $r \neq k$)

$$C([\lambda+\Delta_r+\Delta_k]) - C([\lambda]) = (C([\lambda+\Delta_r]) - C([\lambda])) + (C([\lambda+\Delta_k]) - C([\lambda]))\,. \quad (41)$$

Thus

$$C([\lambda+\Delta_r+\Delta_k]) - C([\lambda+\Delta_k]) = C([\lambda+\Delta_r) - C([\lambda]) \quad (42)$$

so that all such closed loops are consistent. This shows that all the ETPGs are consistent.

With the help of the Young diagram notation we now recast (19) in the form

$$\check{R}(x|\alpha,\beta) = \sum_{[\lambda]}{}' \rho_{[\lambda]}(x) P^{\alpha\beta}_{[\lambda]} P_{[\lambda]} \equiv \check{R}(1|\alpha,\beta) P_{[\lambda]} \quad (43)$$

where the prime signifies the summation over allowed Young diagrams, as in the tensor product decomposition (32). Since the ETPG is consistent we may calculate the coefficients $\rho_{[\lambda]}(x)$ by succesive removal of boxes, starting with the last column and proceeding to eliminate column by column. By this means we arrive at

$$\rho_{[\lambda]}(x) = \prod_{l=1}^{r}\prod_{k=1}^{\lambda_l} \langle 2k-\alpha-\beta-2l\rangle \qquad [\lambda] \equiv [\lambda_1, \lambda_2, \cdots \lambda_r] \quad (44)$$

where we have chosen the normalization $\rho_{.}(x) = 1$ (corresponding to the highest vertex $\Lambda_{\alpha+\beta}$) and the notation $\langle a\rangle$ is as in (22).

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